$Boundary\ value-\ problem$

$The\ method\ of\ Finite\ Difference:$

$The\ derivative\ \frac{dy}{dx}\ is\ the\ ratio\ of\ very\ small$

$(dy)\ over\ very\ small\ (dx).$

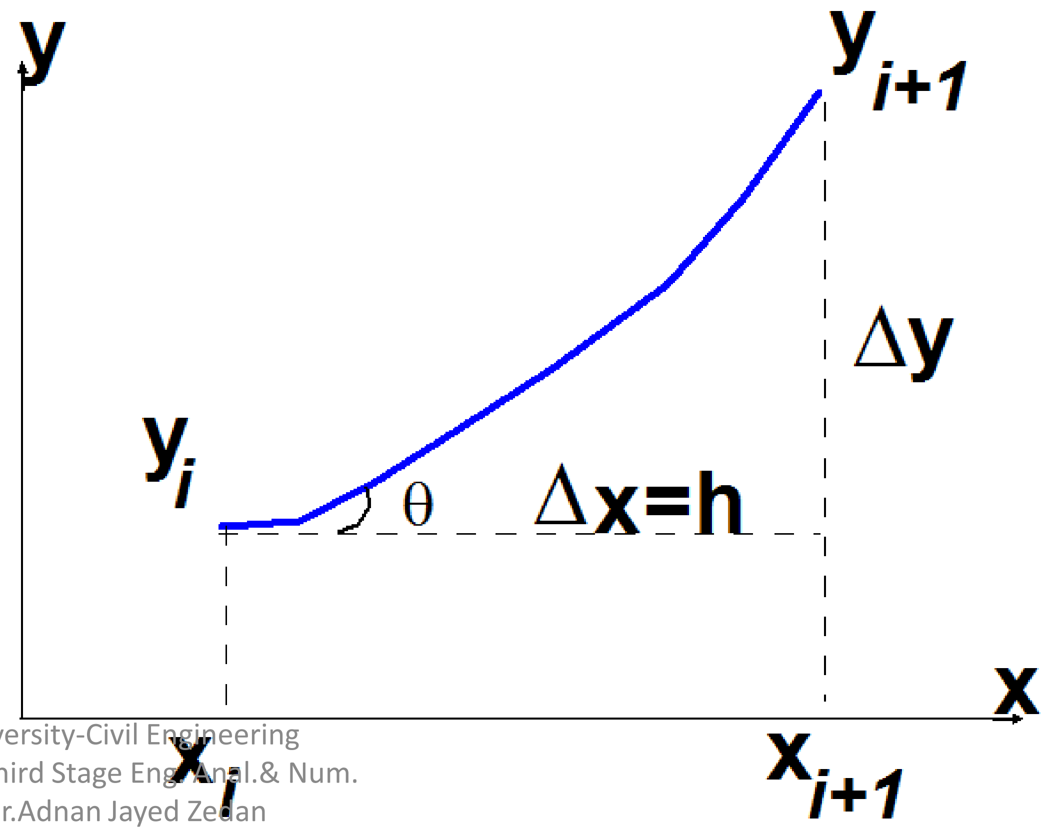
$In\ finite\ difference\ technique,\ the\ derivative \frac{dy}{dx}\ is$

$given\ as\ \frac{\Delta y}{\Delta x}.$

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$(x_{i+1} - x_i = h),\ where;$$

$$(h\ is\ the\ step\ size)$$

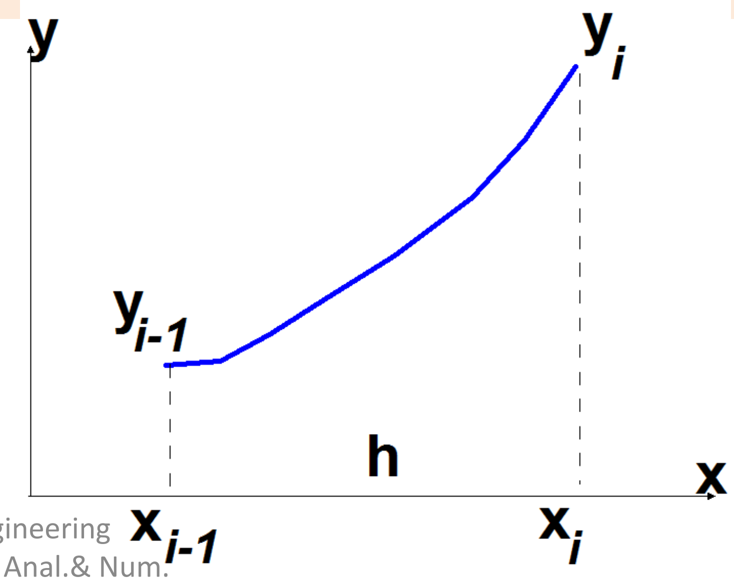


$$1 - Forward - method:$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} \qquad (forward\ \ difference)$$

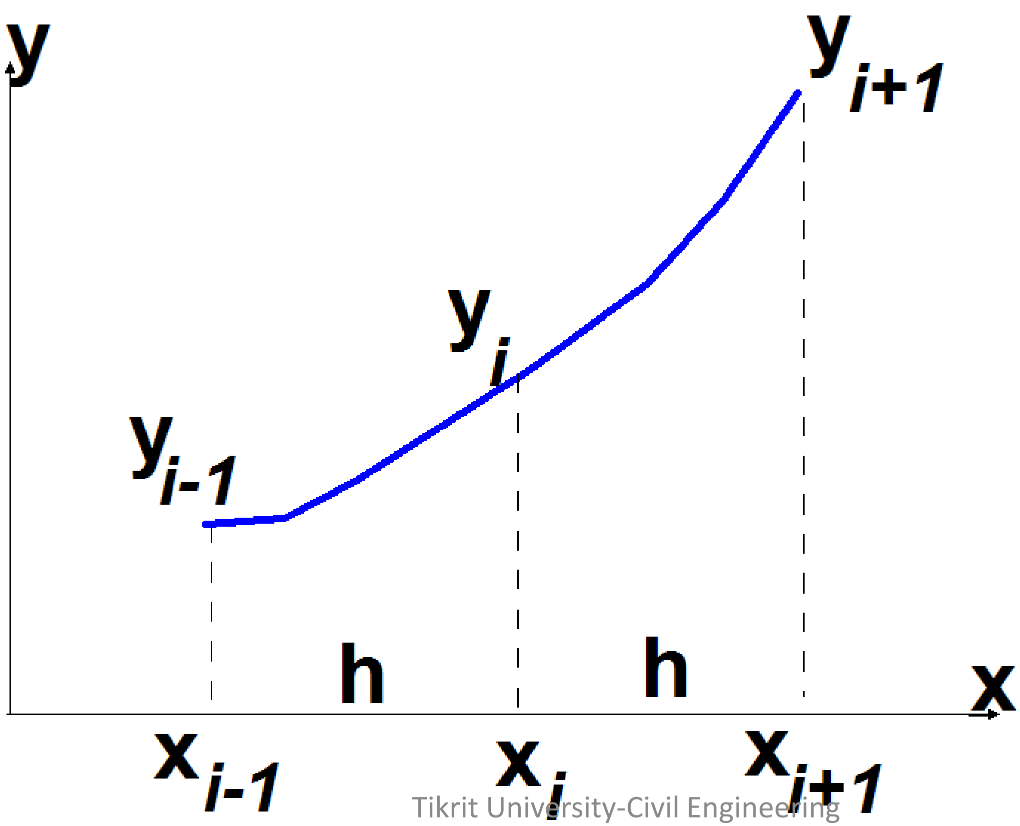
$$2 - Backward - method:$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{y_i - y_{i-1}}{h} \qquad (backward\ \ difference)$$



$\underline{3-Central\ difference}$

$$\frac{dy}{dx}=\frac{\Delta y}{\Delta x}=\frac{y_{i+1}-y_{i-1}}{x_{i+1}-x_{i-1}}=\frac{y_{i+1}-y_{i-1}}{2h} \qquad (central\ \ difference)$$



Second derivative: $\frac{\Delta^2 y}{\Delta x^2}$

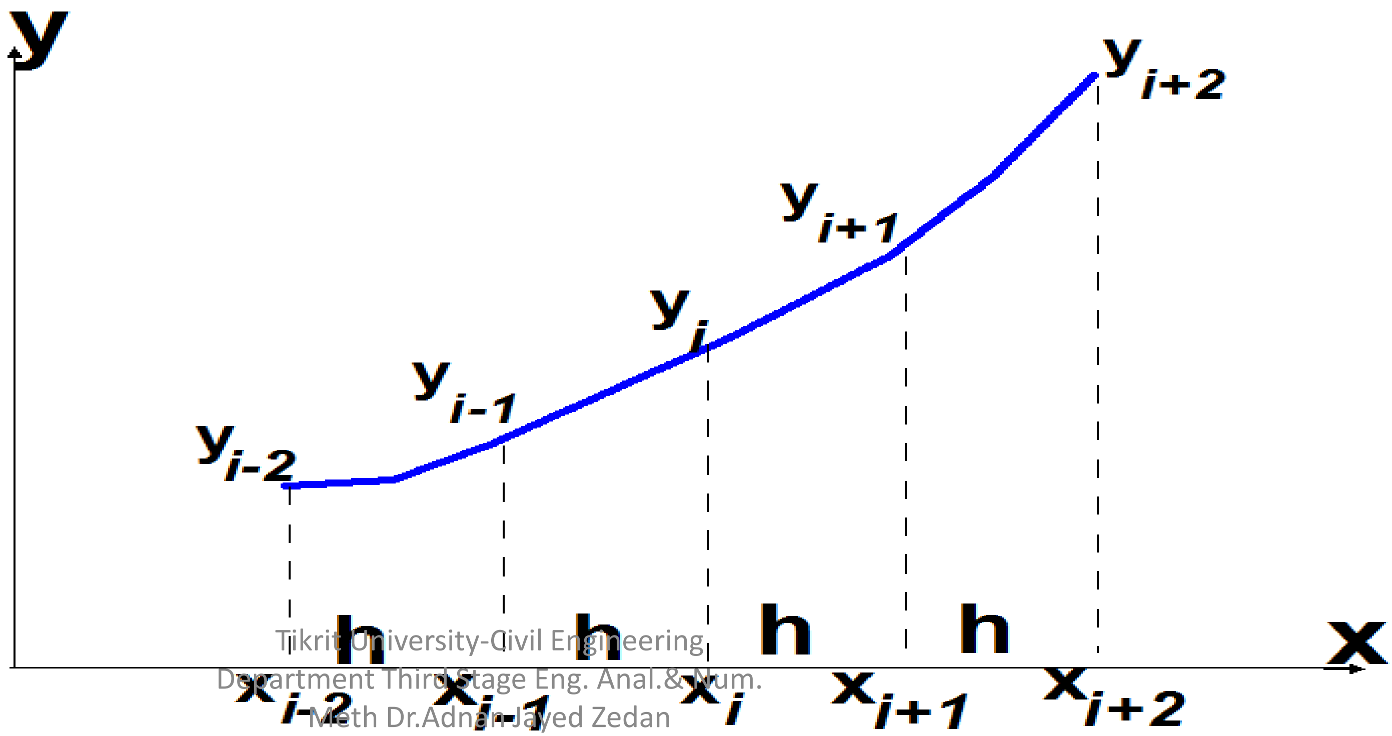
$$\frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{h}$$

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{\left(\frac{y_{i+1} - y_i}{h}\right) - \left(\frac{y_i - y_{i-1}}{h}\right)}{h} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Fourth derivative:

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$\frac{\Delta^4 y}{\Delta x^4} = \frac{\Delta^2}{\Delta x^2}\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right)$$



$$\frac{\Delta^4 y}{\Delta x^4} = \frac{y''_{i+1} - 2y''_i + y''_{i-1}}{h^2}$$

$$= \frac{\left(\dfrac{y_{i+2} - 2y_{i+1} + y_i}{h^2}\right) - 2\left(\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\dfrac{y_i - 2y_{i-1} + y_{i-2}}{h^2}\right)}{h^2}$$

$$\therefore \frac{\Delta^4 y}{\Delta x^4} = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}$$

In Brief :

$$\frac{\Delta y}{\Delta x} = \frac{1}{h}\begin{bmatrix} -1 & +1 \\ y_i & y_{i+1} \end{bmatrix}$$ *forward difference*

$$\frac{\Delta y}{\Delta x} = \frac{1}{h}\begin{bmatrix} -1 & +1 \\ y_{i-1} & y_i \end{bmatrix}$$ *backward difference*

$$\frac{\Delta y}{\Delta x} = \frac{1}{2h}\begin{bmatrix} -1 & +1 \\ y_{i-1} & y_{i+1} \end{bmatrix}$$ *central difference*

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 \\ i-1 & i & i+1 \end{bmatrix}$$

$$\frac{\Delta^4 y}{\Delta x^4} = \frac{1}{h^4}\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ i-2 & i-1 & i & i+1 & i+2 \end{bmatrix}$$

$$\left(\sum Coefficients = 0\right)$$

Application to the solution of differential equations:

The condition required depends on the order of the D.E.:

i) 2^{nd} order D.E. requires two conditions:

These are given at the beginning and at the end of the domain.

ii) 4^{th} order D.E. requires four conditions:

Each two are given at the beginning and at the end of the domain.

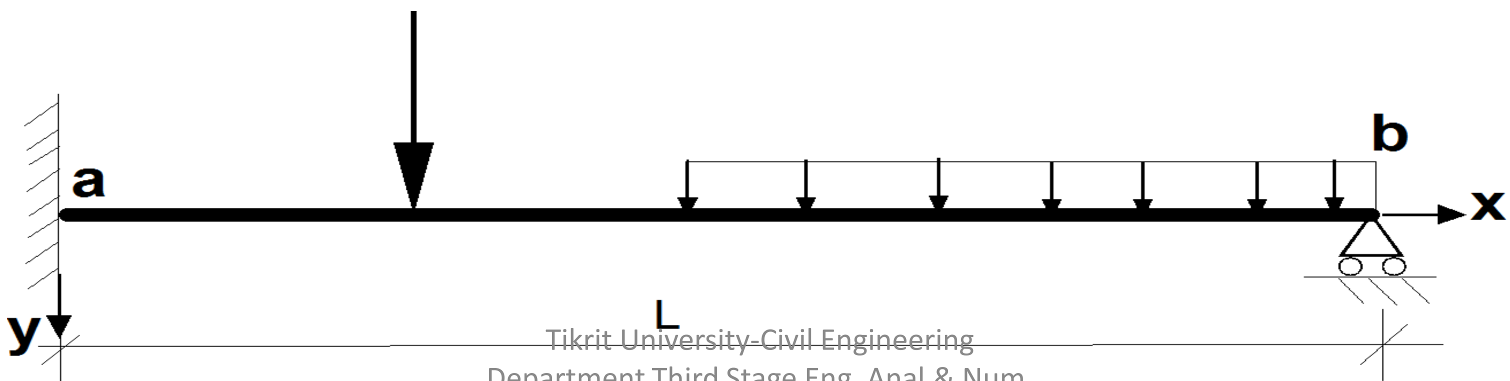
$$\frac{d^4 y}{dx^4} = \frac{q}{EI}$$

$\underline{B.conditions}:$

$$At \quad x = 0 \Rightarrow y = 0 \; and \; \frac{dy}{dx} = 0$$

$$At \quad x = L \Rightarrow y = 0 \; and \; M = 0$$

$$\frac{d^2 y}{dx^2} = \frac{M}{EI} = 0$$



$$B.conditions:$$

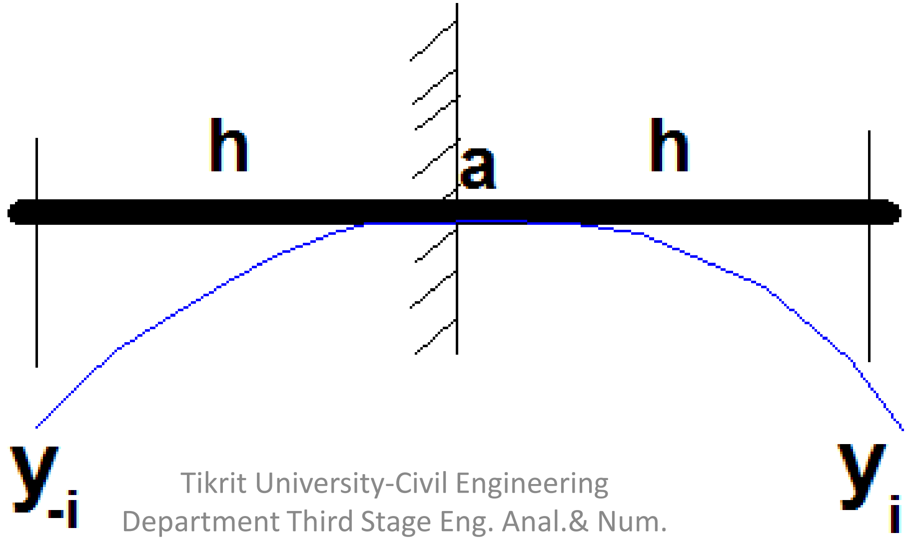
$$At \quad x = 0 \Rightarrow y = 0 \; and \; \frac{dy}{dx} = 0$$

$B.\ conditions\ of\ \ fixed\ end:$

$$Slope\ at\ a = 0$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{2h}\left[-y_{-i} \quad + y_i\right] = 0$$

$$\therefore y_{-i} = y_i$$

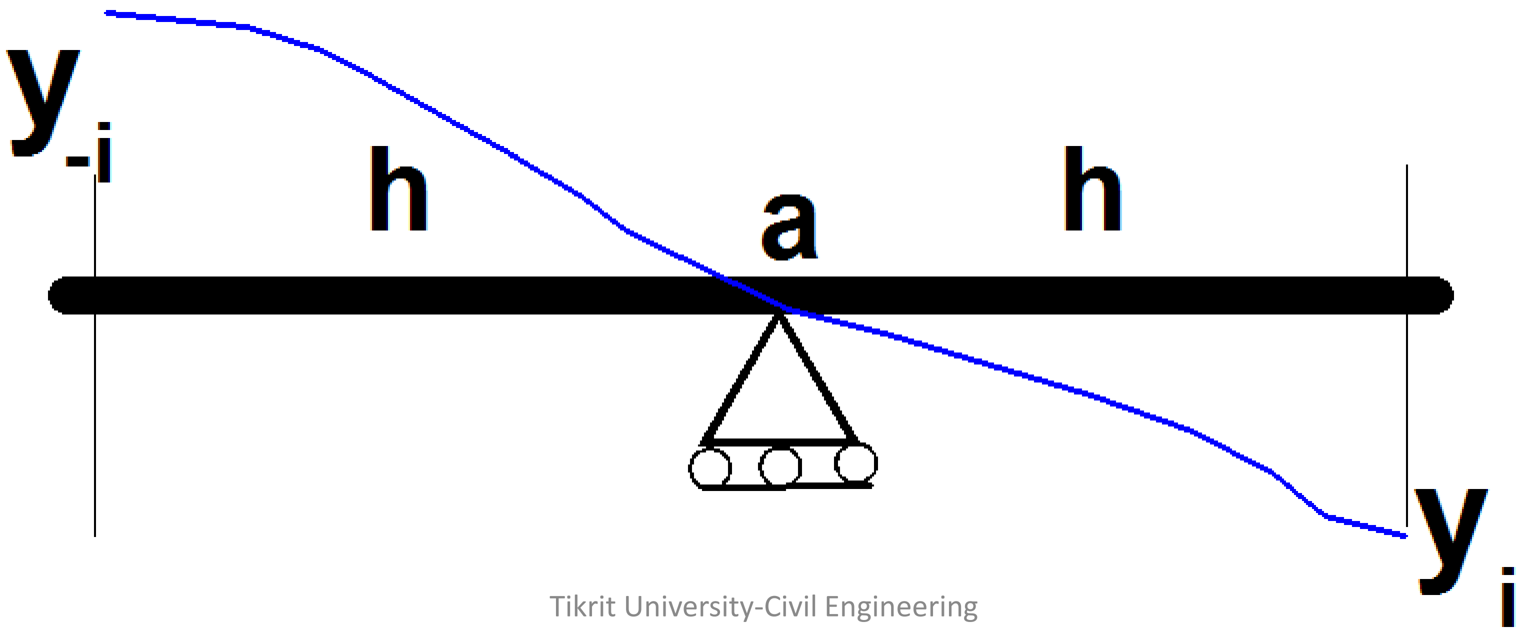


$B.\ conditions\ of\ simply\ supported\ beam:$

$$Moment\ at\ a = 0$$

$$M = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[y_{-i} \quad -2*0 \quad y_i\right] = 0$$

$$\therefore y_{-i} = -y_i$$



$Applications:$

$Example(1):$

$$Solve\ \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x$$

$$y(0) = 0; \quad y(1) = 6.5; \quad h = 0.2$$

Solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x$$

$$\frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1}] - 3\frac{1}{h}[y_i - y_{i-1}] + 2y_i = x_i$$

$For\ \ y_1 = ?\ \ \ i = 1$

$$\frac{1}{(0.2)^2}[0 - 2y_1 + y_2] - 3\frac{1}{0.2}[y_1 - 0] + 2y_1 = 0.2\ \text{......}(1)$$

$For\ \ y_2 = ?\ \ \ i = 2$

$$\frac{1}{(0.2)^2}[y_1 - 2y_2 + y_3] - 3\frac{1}{0.2}[y_2 - y_1] + 2y_2 = 0.4\ \text{......}(2)$$

$For\ \ y_3 = ?\ \ \ i = 3$

$$\frac{1}{(0.2)^2}[y_2 - 2y_3 + y_4] - 3\frac{1}{0.2}[y_3 - y_2] + 2y_3 = 0.6\ \text{......}(3)$$

$For\ \ y_4 = ?\ \ \ i = 4$

$$\frac{1}{(0.2)^2}[y_3 - 2y_4 + 6.5] - 3\frac{1}{0.2}[y_4 - y_3] + 2y_4 = 0.8\ \text{......}(4)$$

$\underline{\underline{H.W.:}}$

$Solve \ for \ y_1, \ y_2, \ y_3 \ and \ y_4 \ using:$

$1 - Gauss - elimination$

$2 - Cramer's \ Rule$

$3 - Matrix \ inverse$

$4 - Iteration$

Example (2):

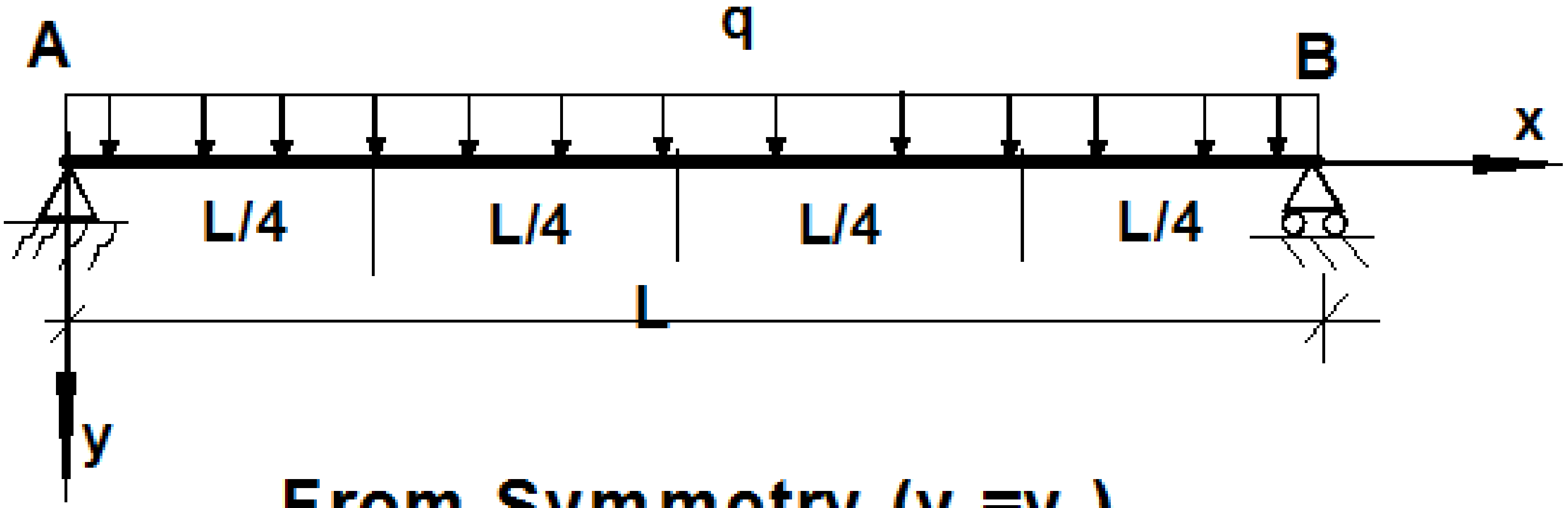
A simply supported beam of length (L)

and constant flexural rigidity [EI],

a uniform load it is carried of (q) per unit length.

Find the deflection, use step size ($h=L/4$).

Solution:



From Symmetry ($y_1 = y_3$)

$$Use\ \frac{d^4y}{dx^4} = \frac{q}{EI}$$

$$\frac{1}{h^4}[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}] = \frac{q_i}{EI}$$

$$For\ \ y_1 = ?\ \ \ ;\ i = 1$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[-y_1 - 4(0) + 6y_1 - 4y_2 + y_1] = \frac{q}{EI}$$

$$6y_1 - 4y_2 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 \ldots\ldots\ldots\ldots(1)$$

$$For\ \ y_2 = ?\ \ ;\ i = 2$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}\left[0 - 4y_1 + 6y_2 - 4y_1 + 0\right] = \frac{q}{EI}$$

$$-8y_1 + 6y_2 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 \ldots\ldots\ldots\ldots(2)$$

$$Solve\ \ Eqs.\,1\&\,2\ \ getting:$$

$$y_1 = \frac{5}{512}\frac{qL^4}{EI}; \quad y_1 = \frac{7}{512}\frac{qL^4}{EI} = 0.013672\frac{qL^4}{EI}$$

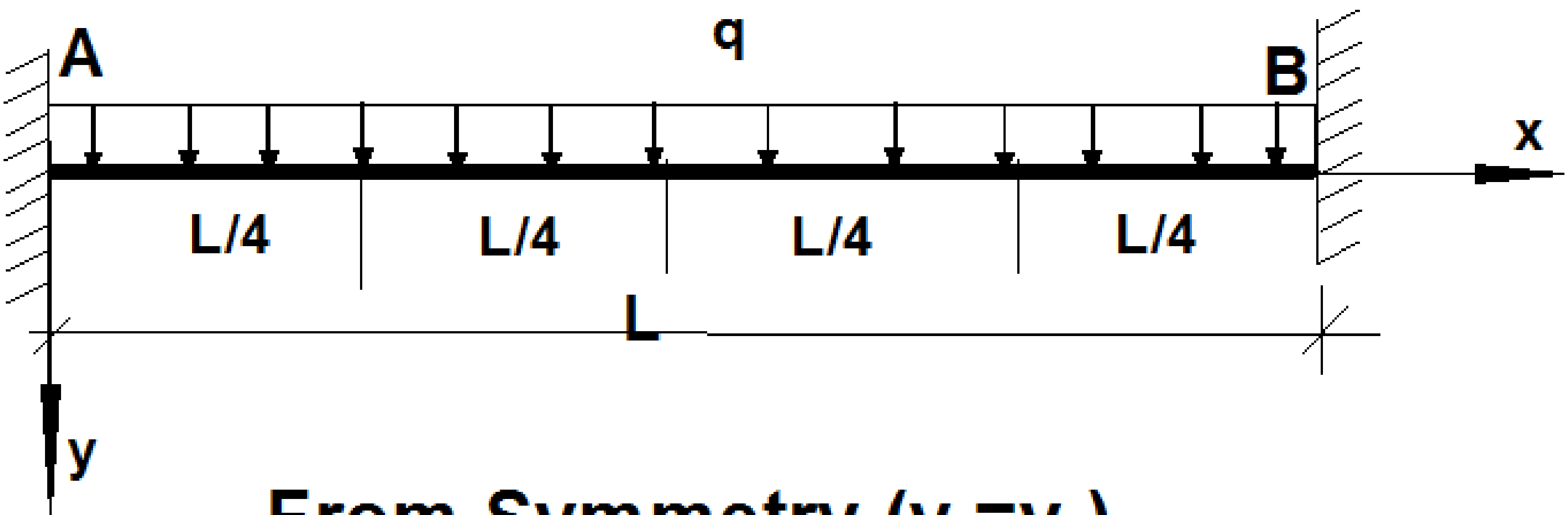
$$y_2\ \ exact = \frac{5qL^4}{384EI} = 0.013021\frac{qL^4}{EI}$$

$$Error \approx 5\%$$

Example (3):

If (A) and (B) are fixed; find the deflections and the fixed moments.

Solution:



From Symmetry ($y_1 = y_3$)

$$Use\ \frac{d^4 y}{dx^4} = \frac{q}{EI}$$

$$\frac{1}{h^4}[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}] = \frac{q}{EI}$$

$$For\ \ y_1 = ?\ \ ;\ i = 1$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}\left[y_1 - 4(0) + 6y_1 - 4y_2 + y_1\right] = \frac{q}{EI}$$

$$8y_1 - 4y_2 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 = A \ldots\ldots\ldots\ldots(1)$$

$$For\ \ y_2 = ?\ \ \ ;i = 2$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[0 - 4y_1 + 6y_2 - 4y_1 + 0] = \frac{q}{EI}$$

$$-8y_1 + 6y_2 = A \quad \ldots\ldots\ldots\ldots(2)$$

$$Solve\ Eqs.\ 1\&2\ getting:$$

$$y_1 = \frac{5}{8}A \qquad ; \quad y_2 = A$$

$$y_1 = \frac{5}{8}\frac{q}{EI}\left(\frac{L}{4}\right)^4 = \frac{5qL^4}{2048EI}$$

$$y_2 = \frac{qL^4}{256EI}$$

$$M_A = ???$$

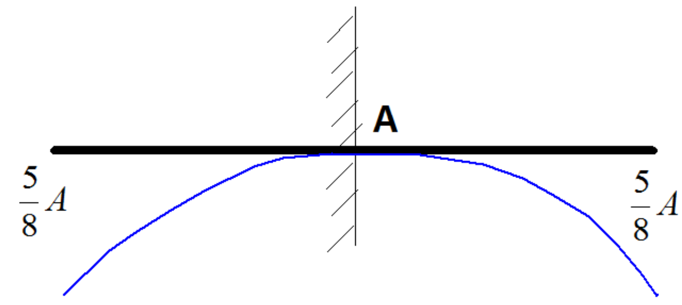
$$\frac{d^2y}{dx^2} = \frac{M}{EI} \Rightarrow M = EI\frac{d^2y}{dx^2}$$

$$M = EI\frac{1}{\left(\frac{L}{4}\right)^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

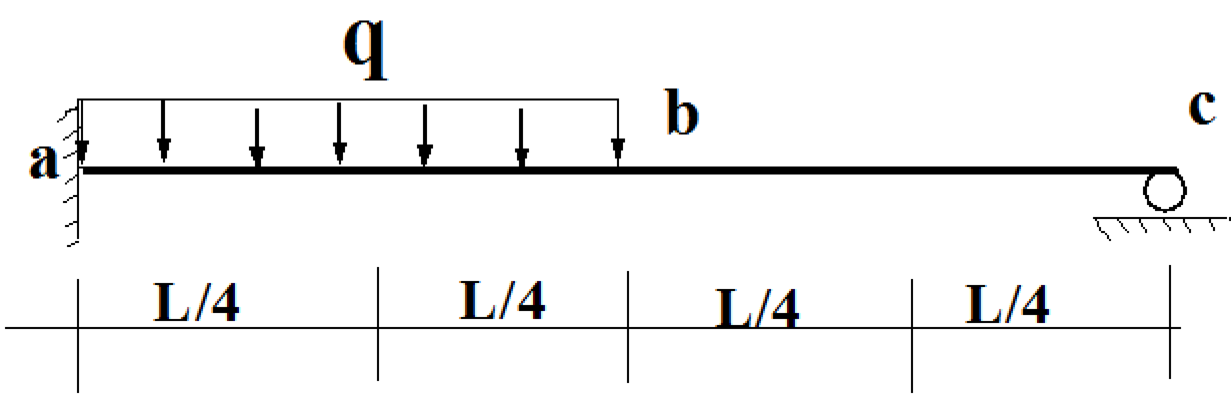
$$= \frac{16EI}{L^2}\left[\frac{5qL^4}{2048EI} - 0 + \frac{5qL^4}{2048EI}\right]$$

$$\therefore M = \frac{160}{2048}qL^2 = 0.078125qL^2$$

$$Exact\ \ M = \frac{qL^2}{12} = 0.0833qL^2$$



$$\underline{\underline{Ex}}: \quad h = \Delta x = \frac{L}{4}; \quad EI \ \text{constant}$$



Solution

$$Use\ \frac{d^4y}{dx^4} = \frac{q}{EI}$$

$$\frac{1}{h^4}[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}] = \frac{q}{EI}$$

$$For\ \ y_1 = ?\quad ; i = 1$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[y_1 - 4(0) + 6y_1 - 4y_2 + y_3] = \frac{q}{EI}$$

$$7y_1 - 4y_2 + y_3 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 = A\ldots\ldots\ldots\ldots(1)$$

$$For\ \ y_2 = ?\ \ ;\ i = 2$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[0 - 4y_1 + 6y_2 - 4y_3 + 0] = \frac{1}{2}\frac{q}{EI}$$

$$-4y_1 + 6y_2 - 4y_3 = \frac{1}{2}A \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$For\ \ y_3 = ?\ \ \ ;\, i = 3$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[y_1 - 4y_2 + 6y_3 - 4(0) - y_3] = 0$$

$$y_1 - 4y_2 + 5y_3 = 0 \quad \ldots\ldots\ldots\ldots(3)$$

$$Solve\ \ Eqs.\,(1),\,(2)\ \&\ (3),\ getting\ y_1,\ y_2\ \&\, y_3$$

$Solution\ of\ \ P.D.E.:$

$Solution\ of\ (2D)\ Steady\ Head\ Flow:$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$$T(x,y)$$

$$\underline{Solution\ of\ \ P.D.E.:}$$

$$\underline{\underline{Solution\ of\ (2D)\ Steady\ Head\ Flow:}}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$$T(x,y)$$

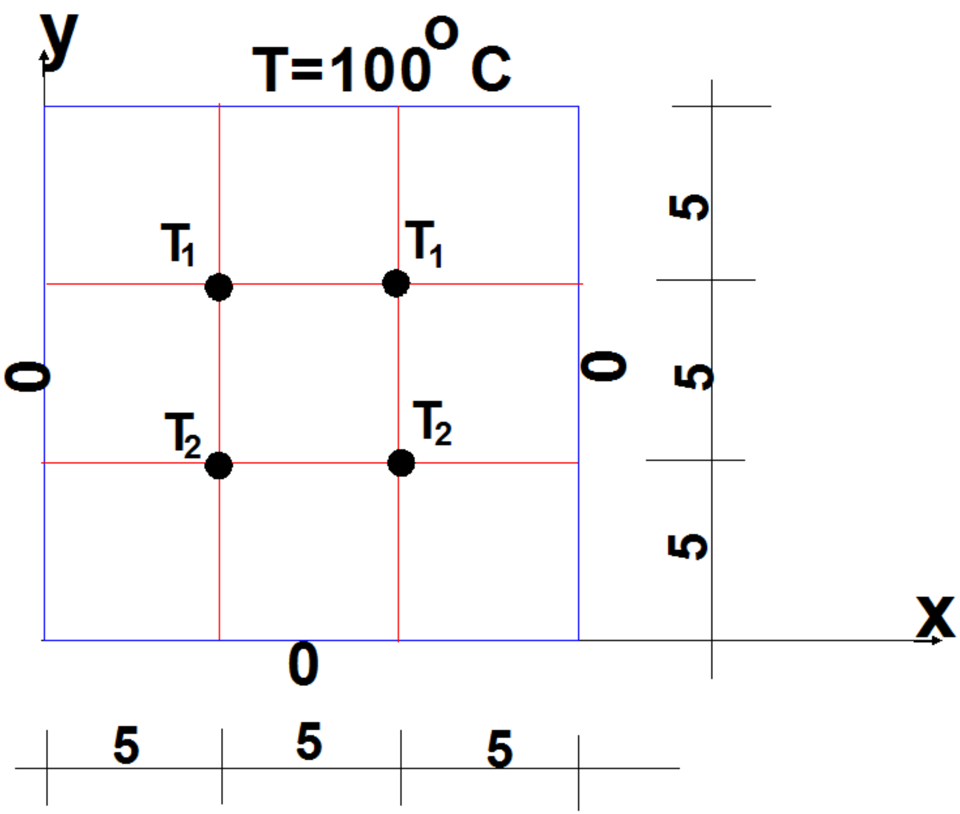
$$\underline{\underline{Boundary\ Conditions:}}$$

$$T(x,15) = 100$$

$$T(x,0) = 0$$

$$T(0,y) = 0$$

$$T(15,y) = 0$$



Grid mesh (3x3)

$$In\ \ general:$$

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{d^2 y}{dx^2} = \frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

$$For\ T_1 = ? \quad i = 1$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{1}{5^2}[0 - 2T_1 + T_1] + \frac{1}{5^2}[T_2 - 2T_1 + 100] = 0 \quad .....(1)$$

$$For\ T_2 = ? \quad i = 2$$

$$\frac{1}{5^2}[0 - 2T_2 + T_2] + \frac{1}{5^2}[0 - 2T_2 + T_1] = 0 \quad .....(2)$$

Solve Eqs. (1) & (2), *getting:*

$$T_1 = 37.5^o C, and\ T_2 = 12.5^o C$$